

# A Dynamical System with Two Strange Attractors.

R. C. Johnson\*

*Department of Mathematical Sciences,  
University of Durham, Durham DH1 3LE, England.*

February 8, 2008

## Abstract

A six-dimensional Rössler-Lorenz hybrid has two coexistent attractors. Both, either or neither may be strange.

**Introduction.** The Rössler [1] and Lorenz [2] equations are textbook examples of chaotic dynamical systems. Each is three-dimensional, continuous-time, smooth and autonomous, with a single strange attractor for certain parameter values.

This paper draws attention to a six-dimensional hybrid with two coexistent attractors, one or both of which may be strange.

The attractors evidently stem from the constituent Rössler and Lorenz systems, but each requires and occupies the larger phase space.

The new system may well have potential for physical modelling, but its discovery is the result of curiosity.

**Construction.** Start with the Lorenz equations [2]

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= (r - z)x - y \\ \dot{z} &= xy - \beta z\end{aligned}\tag{1}$$

in a chaotic regime with

$$\sigma = 10, \quad r = 28, \quad \beta = \frac{8}{3}.\tag{2}$$

---

\*bob.johnson@dur.ac.uk

Fig. 1(a) shows a corresponding solution trajectory on the familiar [3, p. 283] two-winged strange attractor.

Consider replacing  $z$  on the right-hand side in the last component by some external excitation  $w(t)$ :

$$xy - \beta z \rightarrow xy - \beta w. \quad (3)$$

The response may be regular or irregular, depending on  $w$ .

For instance, with  $w = \cos t$ , there is steady oscillation about values

$$x = y = 0, \quad z = r = 28.$$

If instead the excitation  $w(t)$  is an isolated pulse then the response is a sudden excursion of  $z(t)$  followed later by sudden relaxation — each change accompanied by spikes of  $x(t)$  and  $y(t)$ .

Now, the chaotic Rössler system [1]

$$\begin{aligned} \dot{u} &= -v - w \\ \dot{v} &= u + av \\ \dot{w} &= b + (u - c)w \end{aligned} \quad (4)$$

with parameters

$$a = 0.2, \quad b = 0.2, \quad c = 5.7 \quad (5)$$

has solution trajectories on a well-known [3, p. 287] twisted-band strange attractor — as Fig. 1(b) shows.

Evidently its third component  $w(t)$  provides an irregular sequence of isolated pulses. Such a sequence, via eq. (3), provokes the Lorenz  $(x, y, z)$  variables into ragged oscillation about  $(0, 0, r)$ .

This chaotic ringing, however, does not have composite properties. For delay-coordinate reconstruction [4] from  $z(t)$  reveals a Rössler-like twisted-band attractor in three embedding dimensions. Lorenz adds nothing.

Higher-dimensional and richer structure is possible — but needs feedback from Lorenz to Rössler. And one among many options is to mix the oscillations of the  $x$ - $y$  pair into those of  $u$  and  $v$ .

Experiments eventually lead to linear driving of  $v$  by  $x$  and  $y$  — with further linear coupling from  $u$  to  $x$  — as follows:

$$\begin{aligned} \dot{u} &= -v - w \\ \dot{v} &= x + ay \\ \dot{w} &= b + (u - c)w \\ \dot{x} &= \sigma(y - x) + \mu u \\ \dot{y} &= (r - z)x - y \\ \dot{z} &= xy - \beta w. \end{aligned} \quad (6)$$

This is the system of interest. With other parameters fixed, adjustments of  $\mu$  and  $\sigma$  give one or two strange attractors, or a cycle plus a strange attractor, or two cycles, or a single cycle.

**Properties.** If parameters  $r, \sigma, \beta, a, b, c$  take the values in eqs. (2) and (5), phase-space volumes shrink rapidly.

For  $\sigma = 10$ , the system of eq. (6) has two coexisting attractors if  $\mu$  is between about 0 and 13. They have the following properties.

- The two attractors centre approximately around  $u = x = y = 0$  and  $z = r = 28$  but are displaced in  $v$  and  $w$ . See Figs. 2 and 3. Rectangular phase-space boxes just sufficient to enclose them differ in volume by a factor of typically  $10^5$ .
- The larger attractor exists for  $\mu \lesssim 13$ , where it is always strange. Its toadstool-like  $xyz$  projection — Fig. 2(a) — is reminiscent of the Lorenz attractor in Fig. 1(a). For this reason it is labelled  $L$ .
- The smaller one exists for  $\mu \gtrsim 0$  and is strange for  $\mu \lesssim 3.26$  — see Fig. 3. As  $\mu$  increases beyond 3.26 (through 50 and beyond) it alternates between regular — i.e., a cycle — and strange, bifurcating by period-doubling. This parallels the behaviour of the Lorenz attractor as  $r$  increases [5].
- Its twisted-band  $uvw$  projection — Fig. 3(b) — persists with changing  $\mu$ . Because it is similar to the Rössler attractor in Fig. 1(b) this attractor is labelled  $R$ .
- The two attractors have  $v_L < v_R$  and  $w_L > w_R$  and, where they coexist, trajectories move much faster round  $L$  than around  $R$  — see Fig. 4.
- Fig. 4 also shows more variability of period and amplitude on  $R$  than on  $L$ , and trajectories on  $R$  and  $L$  with  $\mu = 1$  are found to have maximum Lyapunov exponent of about 1.4 and 0.45 respectively.
- Both attractors, where they are strange, are higher-dimensional objects. For delay-coordinate reconstruction [4] of  $R$  and  $L$  from samples of components  $u, v, \dots, z$  shows that they need about 5 embedding dimensions.

These properties depend in different ways on  $r, \beta, a, b, c$  and  $\sigma$ .

- A change of  $r$ , for instance, shifts  $R$  and  $L$  bodily along the  $z$ -axis. And while minor variations in  $a, b, c$  and  $\beta$  affect other numerical details, there are larger changes with  $\sigma$ .

- With  $\mu = 1$ ,  $L$  is a cycle for  $2.8 \lesssim \sigma \lesssim 4.5$  — and then  $R$  is also a cycle for  $\sigma \approx 3.7$  — see Fig. 5.
- As Fig. 5(a) suggests, both cycles bifurcate to strange attractors by period-doubling. In this regime they overlap in all but their  $v$  component — they still have  $v_L < v_R$ .

Note that when  $R$  and  $L$  vanish with parameter changes it is because their basins of attraction shrink. No examples have been found where they coalesce.

Thorough exploration of six-dimensional phase space, varying seven parameters, is a formidable task. However, investigation of the  $\mu$ - $\sigma$  plane is promising, and some further illustrations are available elsewhere [6].

**Conclusion.** As it stands, the system of eq. (6) is no more than an interesting construction without direct physical origin. It is just one in a range of possible Rössler-Lorenz hybrids.

Its immediate value is to show that six-dimensional phase space is roomy enough to accommodate two attractors not connected by (e.g.) symmetry — and that all combinations of chaotic and regular oscillation may coexist.

Dynamical models that use in the order of six effective dimensions occur in for instance physiology (e.g. [7]), meteorology (e.g. [8]) and economics (e.g. [9]) — where the possibility of such coexistent modes could be very interesting.

**Acknowledgement.** The trajectory calculations used *XPPAUT* [10]. Lyapunov exponents and embedding dimension were determined with *NDT* [11]. The authors are applauded for the functionality and availability of their software.

## References

- [1] O. E. Rössler, Phys. Lett. A 57 (1976) 397.
- [2] E. N. Lorenz, J. Atmos. Sci. 20 (1963) 130.
- [3] R. H. Abraham and C. D. Shaw, *Dynamics: the Geometry of Behaviour* (Addison-Wesley, Redwood City CA) 2nd ed. 1992.
- [4] H. Kantz and T. Schreiber, *Nonlinear Time-Series Analysis* (CUP, Cambridge) 1997.
- [5] C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors* (Springer, NY) Appl. Math. Sci. 41 (1982).
- [6] *Ro-Lo Home Page*, [maths.dur.ac.uk/~dma0rcj/RoLo/](http://maths.dur.ac.uk/~dma0rcj/RoLo/)
- [7] B. J. West, *Fractal Physiology and Chaos in Medicine*, Studies of Non-linear Phenomena in Life Science Vol. 1 (World Scientific Publishing) 1990;  
C. C. King, *Fractal and Chaotic Dynamics in Nervous Systems*, Progress in Neurobiology 36 (1991) 279.
- [8] The Met.Office, *Numerical Weather Prediction*,  
[www.met-office.gov.uk/sec5/sec5pg2.html](http://www.met-office.gov.uk/sec5/sec5pg2.html)
- [9] J. Benhabib, *On Cycles and Chaos in Economics* (Bibliography),  
[www.mitpress.com/e-journals/SNDE/001/abstracts/v1n1001.html](http://www.mitpress.com/e-journals/SNDE/001/abstracts/v1n1001.html);  
A. Garratt, K. Lee, M. Hashem Pesaran and Y. Shin, *A Long-Run Structural Macroeconometric Model of the UK*, (University of Cambridge DAE Working Paper No.9812, revised 1999),  
[www.econ.cam.ac.uk/faculty/pesaran/ukm95.pdf](http://www.econ.cam.ac.uk/faculty/pesaran/ukm95.pdf)
- [10] G. B. Ermentrout, *XPPAUT*, [www.pitt.edu/~phase/](http://www.pitt.edu/~phase/)
- [11] J. Reiss, *Nonlinear Dynamics Toolbox*,  
[www.physics.gatech.edu/chaos/research/NDTOnline/aboutndt/ndt.htm](http://www.physics.gatech.edu/chaos/research/NDTOnline/aboutndt/ndt.htm)

## Figure captions

**Figure 1** (a) Solution of the Lorenz system eq. (1) with parameters as in eq. (2) — a trajectory on its two-winged strange attractor. (b) Solution of the Rössler system eq. (4) with parameters as in eq. (5) — a trajectory on its twisted-band strange attractor.

**Figure 2** System of eq. (6) at  $\mu = 1$  with other parameters as in eqs. (2) and (5). The  $L$ -attractor — (a)  $xyz$ -projection of a trajectory; (b)  $uvw$ -projection of a trajectory.

**Figure 3** System of eq. (6) at  $\mu = 1$  with other parameters as in eqs. (2) and (5). The  $R$ -attractor — (a)  $xyz$ -projection of a trajectory; (b)  $uvw$ -projection of a trajectory.

**Figure 4** System of eq. (6) at  $\mu = 1$  with other parameters as in eqs. (2) and (5). Component  $z(t)$  of trajectories on (a) the  $L$ -attractor and (b) the  $R$ -attractor. Compare the time scales and variability of period and amplitude.

**Figure 5** System of eq. (6) at  $\mu = 1$  with  $\sigma = 3.7$  and otherwise parameters as in eqs. (2) and (5). (a) The  $L$ -attractor —  $xyz$ -projection of a trajectory. (b) The  $R$ -attractor —  $uvw$ -projection of a trajectory. Both attractors are cycles, and  $L$  is doubled.

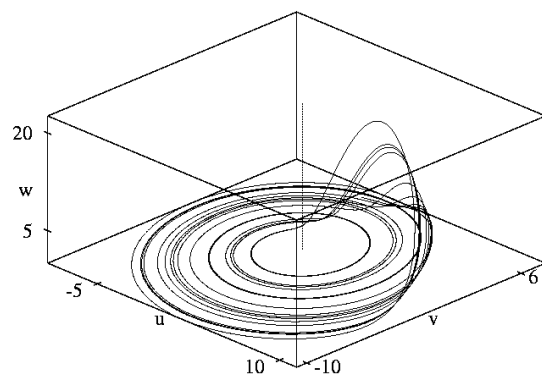
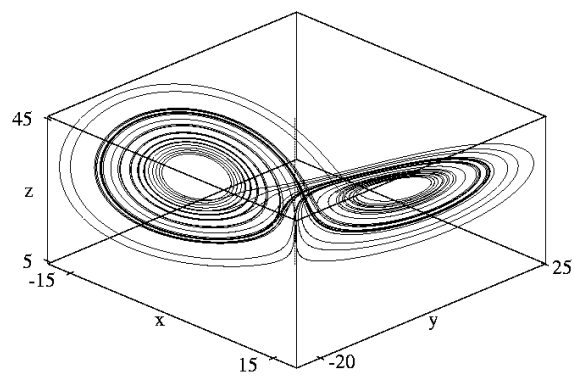


Figure 1: (a) above, (b) below

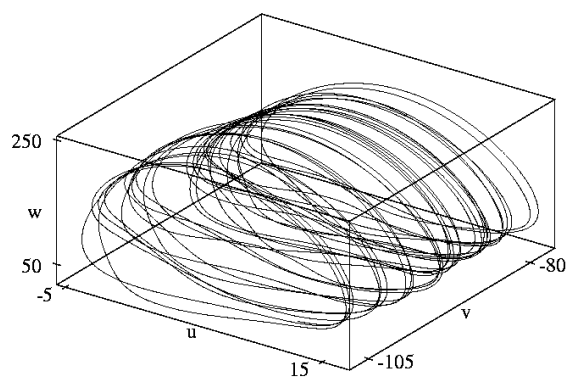
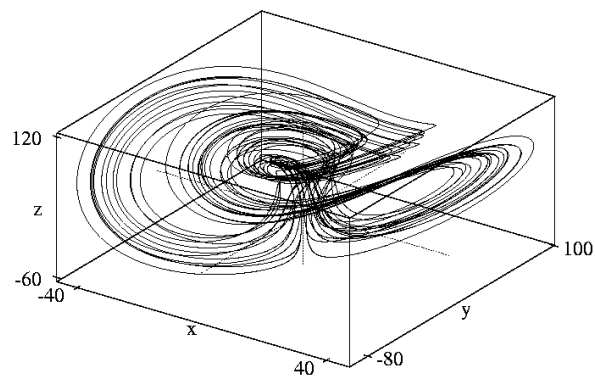


Figure 2: (a) above, (b) below



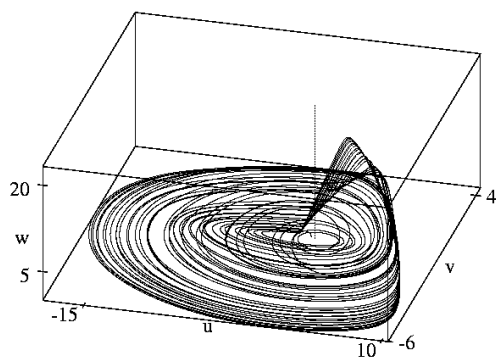
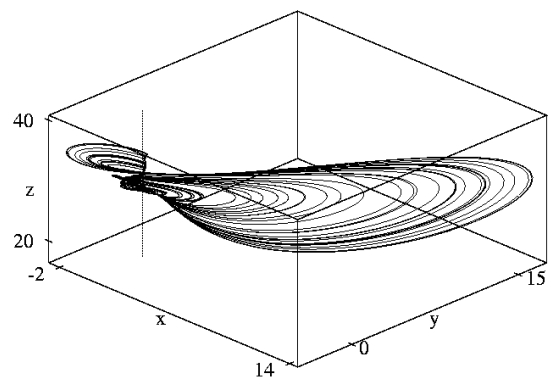


Figure 3: (a) above, (b) below

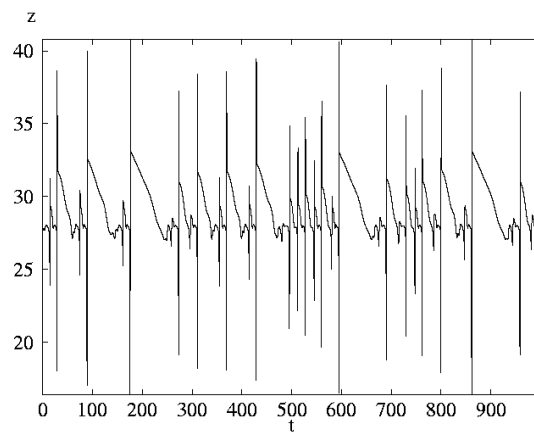
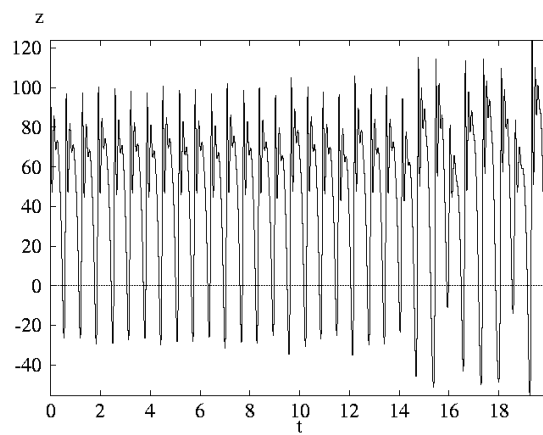


Figure 4: (a) above, (b) below

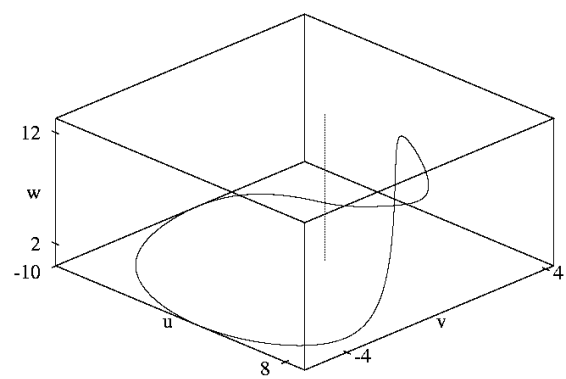
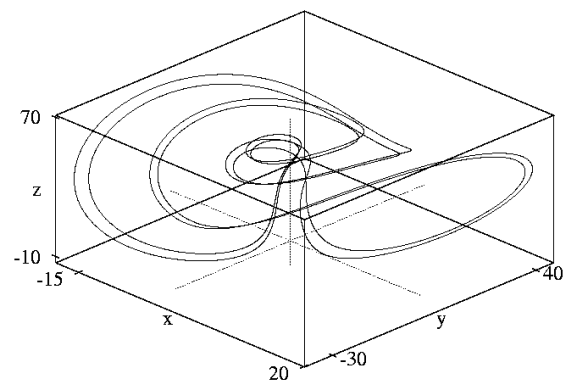


Figure 5: (a) above, (b) below